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Concurrence classes for general pure multipartite states

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Abstract
We propose concurrence classes for general pure multipartite states based on an orthogonal complement of a positive operator-valued measure on quantum phase. In particular, we construct W^m class, GHZ^m, and GHZ^{m-1} class concurrences for general pure m-partite states. We give explicit expressions for W^3 and GHZ^3 class concurrences for general pure three-partite states and for W^4, GHZ^4 and GHZ^3 class concurrences for general pure four-partite states.

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1. Introduction

Entanglement is an interesting feature of quantum theory, which in recent years has attracted many researchers to quantify, classify and to investigate its useful properties. Entanglement has already some applications such as quantum teleportation and quantum key distribution, and there will be new applications for this fascinating quantum phenomenon. For example, multipartite entanglement has the capacity to offer new unimaginable applications in emerging fields of quantum information and quantum computation. One of the widely used measures of entanglement for a pair of qubits is the concurrence, which is directly related to the entanglement of formation [1–3]. In recent years, there have been some proposals to generalize this measure to a general bipartite state. For example, Uhlmann [4] has generalized the concept of concurrence by considering arbitrary conjugation. Later Audenaert *et al* [5] generalized this formula in spirit of Uhlmann's work, by defining a concurrence vector for a pure bipartite state. Another generalization of concurrence was suggested by Rungta *et al* [6] on the basis of a super operator called universal state inversion. Moreover, Gerjuoy [7] and Albeverio and Fei [8] gave an explicit expression of the concurrence in terms of the coefficients of a general pure bipartite state. It would therefore be interesting to be able to generalize this measure from bipartite to a general multipartite state; see [9–11]. Quantifying entanglement of multipartite states has been discussed in [12–24]. In [25, 26], we have proposed a degree of entanglement for a general pure multipartite state based on a positive operator-valued measure

(POVM) on quantum phase. Recently, we have also defined concurrence classes for multi-qubit mixed states [27] based on an orthogonal complement of a POVM on quantum phase. In this paper, we will construct different concurrence classes for general pure multipartite states. Our concurrence classes vanish on the product state by construction. For multi-qubit states, the W^m class concurrences are invariant under stochastic local quantum operation and classical communication (SLOCC) [21], since orthogonal complement of our POVM are invariant under the action of the special linear group. Furthermore, all homogeneous positive functions of pure states that are invariant under determinant-one SLOCC operations are entanglement monotones [23]. However, invariance under SLOCC for the W^m class concurrence for general multipartite states need deeper investigation. It is worth mentioning that Uhlmann [4] has shown that entanglement monotones for concurrence are related to antilinear operators. The GHZ^m class concurrences for multipartite states introduced in this paper are not entanglement monotones except under additional conditions. Thus, the GHZ^m class concurrences need further investigation. Classification of multipartite states has been discussed in [11, 28–31]. For example, Verstraete *et al* [28] have considered a single copy of a pure four-partite state of qubits and investigated its behaviour under SLOCC, which gave a classification of all different classes of pure states of four qubits. They have also shown that there exist nine families of states corresponding to nine different ways of entangling four qubits. Osterloh and Siewert [29] have constructed entanglement measures for pure states of multipartite qubit systems. The key element of their approach is an antilinear operator that they called comb. For qubits, the combs are invariant under the action of the special linear group. They have also discussed inequivalent types of genuine four-qubit entanglement, and found three types of entanglement for these states. This result coincides with our classification, where in section 6 we construct three types of concurrence classes for four-qubit states. Miyake [30] has also discussed classification of multipartite states in entanglement classes based on the determinant. He has shown that two states belong to the same class if they are interconvertible under SLOCC. Moreover, the only paper that addressed the classification of higher-dimensional multipartite states is the paper by Miyake and Verstraete [31], where they have classified multipartite entangled states in the $2 \times 2 \times n$ quantum systems for ($n \geqslant 4$). They have shown that there exist nine essentially different classes of states, and they give rise to a five-graded partially ordered structure, including GHZ class and W class of 3 qubits. Finally, Wang [11] has proposed two classes of the generalized concurrence vectors of the multipartite systems consisting of qubits. Our classification is similar to Wang's classification of multipartite state. However, the advantage of our method is that our POVM can distinguish these concurrence classes without prior information about in equivalence of these classes under local quantum operation and classical communication (LOCC).

Let us denote a general, multipartite quantum system with m subsystems by $\mathcal{Q} = \mathcal{Q}_m(N_1, N_2, \dots, N_m) = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$, consisting of a state

$$|\Psi\rangle = \sum_{k_1=1}^{N_1}\sum_{k_2=1}^{N_2}\cdots\sum_{k_m=1}^{N_m} \alpha_{k_1,k_2,\dots,k_m}|k_1, k_2, \dots, k_m\rangle \tag{1}$$

and, let $\rho_{\mathcal{Q}} = \sum_{n=1}^{\mathrm{N}} p_n|\Psi_n\rangle\langle\Psi_n|$, for all $0 \leqslant p_n \leqslant 1$ and $\sum_{n=1}^{\mathrm{N}} p_n = 1$, denote a density operator acting on the Hilbert space $\mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \cdots \otimes \mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the jth Hilbert space is given by $N_j = \dim\left(\mathcal{H}_{\mathcal{Q}_j}\right)$. Moreover, let us introduce a complex conjugation operator $\mathcal{C}_m$ that acts on a general state $|\Psi\rangle$ of a multipartite state as

$$\mathcal{C}_m|\Psi\rangle = \sum_{k_1=1}^{N_1}\sum_{k_2=1}^{N_2}\cdots\sum_{k_m=1}^{N_m} \alpha^*_{k_1,k_2,\dots,k_m}|k_1, k_2, \dots, k_m\rangle. \tag{2}$$

We are going to use this notation throughout this paper, i.e., we denote a mixed pair of qubits by $\mathcal{Q}_2(2,2)$. The density operator $\rho_{\mathcal{Q}}$ is said to be fully separable, which we will denote by $\rho_{\mathcal{Q}}^{\text{sep}}$, with respect to the Hilbert space decomposition, if it can be written as $\rho_{\mathcal{Q}}^{\text{sep}} = \sum_{n=1}^{N} p_n \bigotimes_{j=1}^{m} \rho_{\mathcal{Q}_j}^{n}$, $\sum_{n=1}^{N} p_n = 1$, for some positive integer N, where p_n are positive real numbers and $\rho_{\mathcal{Q}_j}^{n}$ denote a density operator on a Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$. If $\rho_{\mathcal{Q}}^{p}$ represents a pure state, then the quantum system is fully separable if $\rho_{\mathcal{Q}}^{p}$ can be written as $\rho_{\mathcal{Q}}^{\text{sep}} = \bigotimes_{j=1}^{m} \rho_{\mathcal{Q}_j}$, where $\rho_{\mathcal{Q}_j}$ is a density operator on $\mathcal{H}_{\mathcal{Q}_j}$. If a state is not separable, then it is called an entangled state.

2. Positive operator-valued measure on quantum phase

In this section, we will define a general POVM on quantum phase; see [25]. This POVM is a set of linear operators $\Delta\big(\varphi_{1,2},\ldots,\varphi_{1,N_j},\varphi_{2,3},\ldots,\varphi_{N_j-1,N_j}\big)$ furnishing the probabilities that the measurement of a state ρ_{Q_j} on the Hilbert space $\mathcal{H}_{Q_j}$ is given by

$$p\big(\varphi_{1,2},\ldots,\varphi_{1,N_j},\varphi_{2,3},\ldots,\varphi_{N_j-1,N_j}\big) = \mathrm{Tr}\big(\rho\Delta\big(\varphi_{1,2},\ldots,\varphi_{1,N_j},\varphi_{2,3},\ldots,\varphi_{N_j-1,N_j}\big)\big), \tag{3}$$

where $\big(\varphi_{1,2},\ldots,\varphi_{1,N_j},\varphi_{2,3},\ldots,\varphi_{N_j-1,N_j}\big)$ are the outcomes of the measurement of the quantum phase. This POVM satisfies the following properties, $\Delta\big(\varphi_{1,2},\ldots,\varphi_{1,N_j},\varphi_{2,3},\ldots,\varphi_{N_j-1,N_j}\big)$ is self-adjoint, positive and normalized, that is

$$\overbrace{\int_{2\pi}\cdots\int_{2\pi}}^{\frac{N_j(N_j-1)}{2}} \mathrm{d}\varphi_{1,2}\cdots\mathrm{d}\varphi_{1,N_j}\,\mathrm{d}\varphi_{2,3}\cdots\mathrm{d}\varphi_{N_j-1,N_j}\,\Delta\big(\varphi_{1,2},\ldots,\varphi_{1,N_j},\varphi_{2,3},\ldots,\varphi_{N_j-1,N_j}\big) = \mathcal{I}_{N_j}, \tag{4}$$

where the integral extends over any 2π intervals. A general and symmetric POVM in a single N_j-dimensional Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$ is given by

$$\Delta\big(\varphi_{1_j,2_j},\ldots,\varphi_{1_j,N_j},\varphi_{2_j,3_j},\ldots,\varphi_{N_j-1,N_j}\big) = \sum_{l_j,k_j=1}^{N_j} \mathrm{e}^{\mathrm{i}\varphi_{k_j,l_j}}|k_j\rangle\langle l_j|$$
$$= \begin{pmatrix} 1 & \mathrm{e}^{\mathrm{i}\varphi_{1,2}} & \cdots & \mathrm{e}^{\mathrm{i}\varphi_{1,N_j-1}} & \mathrm{e}^{\mathrm{i}\varphi_{1,N_j}} \\ \mathrm{e}^{-\mathrm{i}\varphi_{1,2}} & 1 & \cdots & \mathrm{e}^{\mathrm{e}\varphi_{2,N_j-1}} & \mathrm{e}^{\mathrm{i}\varphi_{2,N_j}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathrm{e}^{-\mathrm{i}\varphi_{1,N_j-1}} & \mathrm{e}^{-\mathrm{i}\varphi_{2,N_j-1}} & \cdots & 1 & \mathrm{e}^{\mathrm{i}\varphi_{N_j-1,N_j}} \\ \mathrm{e}^{-\mathrm{i}\varphi_{1,N_j}} & \mathrm{e}^{-\mathrm{i}\varphi_{2,N_j}} & \cdots & \mathrm{e}^{-\mathrm{i}\varphi_{N_j-1,N_j}} & 1 \end{pmatrix}, \tag{5}$$

where $|k_j\rangle$ and $|l_j\rangle$ are the basis vectors in $\mathcal{H}_{\mathcal{Q}_j}$ and the quantum phases satisfy the following relation $\varphi_{k_j,l_j} = -\varphi_{l_j,k_j}\big(1-\delta_{k_jl_j}\big)$. The POVM is a function of the $N_j(N_j-1)/2$ phases $\big(\varphi_{1_j,2_j},\ldots,\varphi_{1_j,N_j},\varphi_{2_j,3_j},\ldots,\varphi_{N_j-1,N_j}\big)$. It is now possible to form a POVM of a multipartite system by simply forming the tensor product

$$\Delta_{\mathcal{Q}}\big(\varphi_{\mathcal{Q}_1;k_1,l_1},\ldots,\varphi_{\mathcal{Q}_m;k_m,l_m}\big) = \Delta_{\mathcal{Q}_1}\big(\varphi_{\mathcal{Q}_1;k_1,l_1}\big)\otimes\cdots\otimes\Delta_{\mathcal{Q}_m}\big(\varphi_{\mathcal{Q}_m;k_m,l_m}\big), \tag{6}$$

where, e.g., $\varphi_{\mathcal{Q}_1;k_1,l_1}$ is the set of POVMs phase associated with subsystems $\mathcal{Q}_1$, for all $k_1, l_1 = 1, 2, \ldots, N_1$, where we need only to consider when $l_1 > k_1$.

3. Concurrence for general pure bipartite states

The concurrence of two-qubit states is defined as $\mathcal{C}(\Psi) = |\langle\Psi|\widetilde{\Psi}\rangle|$, where the tilde represents the 'spin-flip' operation $|\widetilde{\Psi}\rangle = \sigma_y\otimes\sigma_y|\Psi^*\rangle$, $|\Psi^*\rangle = \sum_{l,k=1}^{2}\alpha_{k,l}^{*}|k,l\rangle$ is the complex conjugate of $|\Psi\rangle = \sum_{l,k=1}^{2} \alpha_{k,l}|k,l\rangle$, and $\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ is a Pauli spin-flip operator [2, 3]. Now, we will define concurrence for a general pure bipartite state based on the orthogonal complement of our POVM by constructing an antilinear operator for a general pure bipartite state $\mathcal{Q}_2^p(N_1, N_2)$. The POVM for quantum system $\mathcal{Q}_2^p(N_1, N_2)$ is given by

$$\Delta_{\mathcal{Q}}\left(\varphi_{\mathcal{Q}_1;k_1,l_1}, \varphi_{\mathcal{Q}_2;k_2,l_2}\right) = \Delta_{\mathcal{Q}_1}\left(\varphi_{\mathcal{Q}_1;k_1,l_1}\right) \otimes \Delta_{\mathcal{Q}_2}\left(\varphi_{\mathcal{Q}_2;k_2,l_2}\right). \tag{7}$$

Next, we define the orthogonal complement of our POVM by $\widetilde{\Delta}_{\mathcal{Q}_j}\left(\varphi_{\mathcal{Q}_j;k_j,l_j}\right) = \mathcal{I}_{N_j} - \Delta_{\mathcal{Q}_j}\left(\varphi_{\mathcal{Q}_j;k_j,l_j}\right)$, where $\mathcal{I}_{N_j}$ is the N_j-by-N_j identity matrix, for each subsystem j. For example, for a bipartite state $\mathcal{Q}_2^p(2,3)$ we have

$$\begin{aligned}\widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi_{\mathcal{Q}_2;k_2,l_2}\right) &= \widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi_{\mathcal{Q}_2;1,2}\right) + \widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi_{\mathcal{Q}_2;1,3}\right) + \widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi_{\mathcal{Q}_2;2,3}\right) \\ &= \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\varphi_{\mathcal{Q}_2;1,2}} & \mathrm{e}^{\mathrm{i}\varphi_{\mathcal{Q}_2;1,3}} \\ \mathrm{e}^{-\mathrm{i}\varphi_{\mathcal{Q}_2;1,2}} & 0 & \mathrm{e}^{-\mathrm{i}\varphi_{\mathcal{Q}_2;2,3}} \\ \mathrm{e}^{-\mathrm{i}\varphi_{\mathcal{Q}_2;1,3}} & \mathrm{e}^{-\mathrm{i}\varphi_{\mathcal{Q}_2;2,3}} & 0 \end{pmatrix}\end{aligned} \tag{8}$$

where, e.g.,

$$\widetilde{\Delta}_{\mathcal{Q}_2}(\varphi_{\mathcal{Q}_2;1,3}) = \begin{pmatrix} 0 & 0 & \mathrm{e}^{\mathrm{i}\varphi_{\mathcal{Q}_2;1,3}} \\ 0 & 0 & 0 \\ \mathrm{e}^{-\mathrm{i}\varphi_{\mathcal{Q}_2;1,3}} & 0 & 0 \end{pmatrix}.$$

Moreover, we have

$$\widetilde{\Delta}_{\mathcal{Q}_1}(\varphi_{\mathcal{Q}_1;1,2}) = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\varphi_{\mathcal{Q}_1;1,2}} \\ \mathrm{e}^{-\mathrm{i}\varphi_{\mathcal{Q}_1;1,2}} & 0 \end{pmatrix}.$$

Then for a quantum system $\mathcal{Q}_2^p(2,3)$ the orthogonal complement of our POVM $\Delta_{\mathcal{Q}}\left(\varphi_{\mathcal{Q}_1;k_1,l_1}, \varphi_{\mathcal{Q}_2;k_2,l_2}\right)$ is given by

$$\begin{aligned}\widetilde{\Delta}_{\mathcal{Q}}\left(\varphi_{\mathcal{Q}_1;k_1,l_1}, \varphi_{\mathcal{Q}_2;k_2,l_2}\right) &= \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi_{\mathcal{Q}_1;k_1,l_1}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi_{\mathcal{Q}_2;k_2,l_2}\right) \\ &= \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi_{\mathcal{Q}_1;1,2}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi_{\mathcal{Q}_2;1,2}\right) + \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi_{\mathcal{Q}_1;1,2}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi_{\mathcal{Q}_2;1,3}\right) \\ &\quad + \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi_{\mathcal{Q}_1;1,2}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi_{\mathcal{Q}_2;2,3}\right).\end{aligned} \tag{9}$$

Now, we will introduce the following notation

$$\widetilde{\Delta}^{\mathrm{EPR}_{k_1,l_1;k_2,l_2}}_{\mathcal{Q}_{1,2}(2,3)} = \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1;1,2}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2;k_2,l_2}\right), \tag{10}$$

where by choosing $\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_j;k_j,l_j} = \frac{\pi}{2}$ for all $k_j < l_j$, $j = 1, 2$, we get an operator which has the structure of Pauli spin-flip operator σ_y embedded in a higher-dimensional Hilbert space and coincides with σ_y for a single qubit. Moreover, EPR indicates that this operator detects the generic bipartite entangled state. We then define the concurrence of quantum system $\mathcal{Q}_2^p(2,3)$ as

$$\begin{aligned}\mathcal{C}\left(\mathcal{Q}_2^p(2,3)\right) &= \left(\sum_{\forall k_1,l_1,k_2,l_2} \left|\left\langle\Psi\right|\widetilde{\Delta}^{\mathrm{EPR}_{k_1,l_1;k_2,l_2}}_{\mathcal{Q}_{1,2}(2,3)}\mathcal{C}_2\Psi\right\rangle\right|^2\right)^{1/2} \\ &= \left(\left|\left\langle\Psi\right|\widetilde{\Delta}^{\mathrm{EPR}_{1,2;1,2}}_{\mathcal{Q}_{1,2}(2,3)}\mathcal{C}_2\Psi\right\rangle\right|^2 + \left|\left\langle\Psi\right|\widetilde{\Delta}^{\mathrm{EPR}_{1,2;1,3}}_{\mathcal{Q}_{1,2}(2,3)}\mathcal{C}_2\Psi\right\rangle\right|^2 + \left|\left\langle\Psi\right|\widetilde{\Delta}^{\mathrm{EPR}_{1,2;2,3}}_{\mathcal{Q}_{1,2}(2,3)}\mathcal{C}_2\Psi\right\rangle\right|^2\right)^{1/2} \\ &= \left(4\mathcal{N}_2^{\mathrm{EPR}}[|\alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1}|^2 + |\alpha_{1,1}\alpha_{2,3} - \alpha_{1,3}\alpha_{2,1}|^2 \right. \\ &\quad \left. + |\alpha_{1,2}\alpha_{2,3} - \alpha_{1,3}\alpha_{2,2}|^2]\right)^{1/2}.\end{aligned} \tag{11}$$

Now, the generalization of this result is straightforward. Hence, for a pure quantum system $\mathcal{Q}_2^p(N_1, N_2)$ we have

$$\widetilde{\Delta}^{\mathrm{EPR}_{k_1,l_1;k_2,l_2}}_{\mathcal{Q}_{1,2}(N_1,N_2)} = \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1;k_1,l_1}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2;k_2,l_2}\right), \tag{12}$$

and the concurrence is given by

$$\mathcal{C}\left(\mathcal{Q}_2^p(N_1, N_2)\right) = \left(\sum_{\forall k_1,l_1,k_2,l_2} \left|\left\langle\Psi\middle|\widetilde{\Delta}^{\mathrm{EPR}_{k_1,l_1;k_2,l_2}}_{\mathcal{Q}_{1,2}(N_1,N_2)}\mathcal{C}_2\Psi\right\rangle\right|^2\right)^{1/2}$$
$$= \left(4\mathcal{N}_2^{\mathrm{EPR}} \sum_{l_1>k_1=1}^{N_1} \sum_{l_2>k_2=1}^{N_2} \left|\alpha_{k_1,k_2}\alpha_{l_1,l_2} - \alpha_{k_1,l_2}\alpha_{l_1,k_2}\right|^2\right)^{1/2}. \tag{13}$$

The concurrence $\mathcal{C}\left(\mathcal{Q}_2^p(N_1, N_2)\right)$ vanishes for product states, and coincides with our entanglement tensor for general bipartite state [26] and with the generalized concurrence given in [7, 8]. Moreover, the concurrence $\mathcal{C}\left(\mathcal{Q}_2^p(N_1, N_2)\right)$ coincides with I-concurrence, which is a generalization of concurrence introduced by Rungta *et al* [6] based on universal state inversion. Furthermore, our antilinear operator $\widetilde{\Delta}^{\mathrm{EPR}_{k_1,l_1;k_2,l_2}}_{\mathcal{Q}_{1,2}(N_1,N_2)}\mathcal{C}_2$ is invariant under the LOCC operation by construction.

4. Concurrence classes for general pure multipartite states

In this section, we will construct concurrence classes for general pure multipartite states $\mathcal{Q}_m^p(N_1, \ldots, N_m)$. In order to simplify our presentation, we will use $\Lambda_m = k_1, l_1; \ldots; k_m, l_m$ as an abstract multi-index notation. The unique structure of our POVM enables us to distinguish different classes of multipartite states, which are inequivalent under LOCC operations. In the m-partite case, the off-diagonal elements of the matrix corresponding to

$$\widetilde{\Delta}_{\mathcal{Q}}\left(\varphi_{\mathcal{Q}_1;k_1,l_1}, \ldots, \varphi_{\mathcal{Q}_m;k_m,l_m}\right) = \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi_{\mathcal{Q}_1;k_1,l_1}\right) \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_m}\left(\varphi_{\mathcal{Q}_m;k_m,l_m}\right), \tag{14}$$

have phases that are sum or differences of phases originating from two and m subsystems. That is, in the later case the phases of $\widetilde{\Delta}_{\mathcal{Q}}\left(\varphi_{\mathcal{Q}_1;k_1,l_1}, \ldots, \varphi_{\mathcal{Q}_m;k_m,l_m}\right)$ take the form $\left(\varphi_{\mathcal{Q}_1;k_1,l_1} \pm \varphi_{\mathcal{Q}_2;k_2,l_2} \pm \cdots \pm \varphi_{\mathcal{Q}_m;k_m,l_m}\right)$ and identification of these joint phases makes our classification possible. Thus, we can define linear operators for the W^m class based on our POVM which are sum and difference of phases of two subsystems, i.e., $\left(\varphi_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}} \pm \varphi_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}}\right)$. That is, for the W^m class we have

$$\widetilde{\Delta}^{W^m_{\Lambda_m}}_{\mathcal{Q}_{r_1,r_2}(N_{r_1},N_{r_2})} = \mathcal{I}_{N_1} \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_1}}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}}\right) \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_2}}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}}\right) \otimes \cdots \otimes \mathcal{I}_{N_m}. \tag{15}$$

Let $C(m, k) = \binom{m}{k}$ denote the binomial coefficient. Then there are $C(m, 2)$ linear operators for the W^m class and the set of these operators gives the W^m class concurrence.

For the GHZ^m class, we define the linear operators based on our POVM which are sum and difference of phases of m-subsystems, i.e., $\left(\varphi_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}} \pm \varphi_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}} \pm \cdots \pm \varphi_{\mathcal{Q}_m;k_m,l_m}\right)$. That is, for the GHZ^m class we have

$$\widetilde{\Delta}^{\mathrm{GHZ}^m_{\Lambda_m}}_{\mathcal{Q}_{r_1,r_2}(N_{r_1},N_{r_2})} = \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi^{\pi}_{\mathcal{Q}_1;k_1,l_1}\right) \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_1}}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}}\right) \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_2}}$$
$$\times \left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}}\right) \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_m}\left(\varphi^{\pi}_{\mathcal{Q}_m;k_m,l_m}\right), \tag{16}$$

where by choosing $\varphi^{\pi}_{\mathcal{Q}_j;k_j,l_j} = \pi$ for all $k_j < l_j$, $j = 1, 2, \ldots, m$, we get an operator which has the structure of Pauli operator σ_x embedded in a higher-dimensional Hilbert space and coincides with σ_x for a single qubit. There are $C(m, 2)$ linear operators for the GHZ^m class and the set of these operators gives the GHZ^m class concurrence.

Moreover, we define the linear operators for the GHZ^{m-1} class of m-partite states based on our POVM which are sum and difference of phases of $m-1$-subsystems, i.e., $\left(\varphi_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}} \pm \varphi_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}} \pm \cdots \varphi_{\mathcal{Q}_{m-1};k_{m-1},l_{m-1}} \pm \varphi_{\mathcal{Q}_{m-1};k_{m-1},l_{m-1}}\right)$. That is, for the GHZ^{m-1} class we have

$$\widetilde{\Delta}^{\mathrm{GHZ}^{m-1}_{\Lambda_m}}_{\mathcal{Q}_{r_1 r_2, r_3}(N_{r_1},N_{r_2})} = \widetilde{\Delta}_{\mathcal{Q}_{r_1}}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_{r_1};k_{r_1},l_{r_1}}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_2}}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_{r_2};k_{r_2},l_{r_2}}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_{r_3}}\left(\varphi^{\pi}_{\mathcal{Q}_{r_3};k_{r_3},l_{r_3}}\right) \otimes \cdots \otimes \widetilde{\Delta}_{\mathcal{Q}_{m-1}}\left(\varphi^{\pi}_{\mathcal{Q}_{m-1};k_{r_{m-1}},l_{r_{m-1}}}\right) \otimes \mathcal{I}_{N_m}, \quad (17)$$

where $1 \leqslant r_1 < r_2 < \cdots < r_{m-1} < m$. There are $C(m, m-1)$ such operators for the GHZ^{m-1} class.

Now, we can construct concurrence classes for multipartite states. For example, let X^m denote two different classes of general multipartite states, namely W^m and GHZ^m classes. Then, for general pure quantum system $\mathcal{Q}^p_m(N_1, \ldots, N_m)$ with

$$\mathcal{C}\left(\mathcal{Q}^{X^m}_{r_1,r_2}\left(N_{r_1}, N_{r_2}\right)\right) = \sum_{\forall k_1,l_1,\ldots,k_m,l_m} \left|\left\langle \Psi \middle| \widetilde{\Delta}^{X^m_{\Lambda_m}}_{\mathcal{Q}_{r_1,r_2}(N_{r_1},N_{r_2})} \mathcal{C}_m \Psi \right\rangle\right|^2, \quad (18)$$

the X^m class concurrences are given by

$$\mathcal{C}\left(\mathcal{Q}^{X^m}_m(N_1, \ldots, N_m)\right) = \left(\mathcal{N}^X_m \sum_{r_2>r_1=1}^{m} \mathcal{C}\left(\mathcal{Q}^{X^m}_{r_1,r_2}\left(N_{r_1}, N_{r_2}\right)\right)\right)^{1/2}, \quad (19)$$

where $\mathcal{N}^X_m$ is a normalization constant. Note that for m-partite states the W^m class concurrences are zero only for completely separable states. That is as long as we have bipartite entanglement in our state this measure does not vanish. Thus, this class also includes all biseparable states. We will discuss this issue in detail in a forthcoming paper. One can say that the W^m class concurrence measures the amount of bipartite entanglement in a multipartite state. Now, let us address the monotonicity of these concurrence classes of multipartite states. For m-qubit states, the W^m class concurrences are entanglement monotones. Let $A_j \in SL(2, \mathbf{C})$, for $j = 1, 2, \ldots, m$, and $\mathcal{A} = A_1 \otimes A_2 \otimes \cdots \otimes A_m$, then $\mathcal{A}\widetilde{\Delta}^{W^m_{1,2;\ldots;1,2}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}\mathcal{A}^T = \widetilde{\Delta}^{W^m_{1,2;\ldots;1,2}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}$, for all $1 < r_1 < r_2 < m$. Thus, the W^m class concurrences for multi-qubit states are invariant under SLOCC, and hence are entanglement monotones. Again, for general multipartite states we cannot give any proof on invariance of the W^m class concurrence under SLOCC and this question needs further investigation. Moreover, for multipartite states, the GHZ^m class concurrences are not entanglement monotone except under additional conditions. Since $\mathcal{A}\widetilde{\Delta}^{\mathrm{GHZ}^m_{1,2;\ldots;1,2}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}\mathcal{A}^T \neq \widetilde{\Delta}^{\mathrm{GHZ}^3_{1,2;\ldots;1,2}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}$, for all $1 < r_1 < r_2 < m$. The reason is that $A_j\widetilde{\Delta}_{\mathcal{Q}_j}\left(\varphi^{\pi}_{\mathcal{Q}_j;1,2}\right)A_j^T \neq \widetilde{\Delta}_{\mathcal{Q}_j}\left(\varphi^{\pi}_{\mathcal{Q}_j;1,2}\right)$. Thus, the GHZ^m class concurrence for three-qubit states are not invariant under SLOCC, and hence are not entanglement monotones. However, by construction the GHZ^m class concurrences are invariant under all permutations. Moreover, we have $\left(\widetilde{\Delta}^{\mathrm{GHZ}^3_{1,2;\ldots;1,2}}_{\mathcal{Q}_{r_1,r_2}(2_{r_1},2_{r_2})}\right)^2 = 1$ and $\left(\widetilde{\Delta}_{\mathcal{Q}_j}\left(\varphi^{\pi}_{\mathcal{Q}_j;1,2}\right)\right)^2 = 1$. Furthermore, we need to be very careful when we are using the GHZ^m class concurrences. This class can be zero even for an entangled multipartite state. Since we have more than two joint phases in our POVM for GHZ^m class concurrence. Thus, for the GHZ^m class concurrences we need to perform an optimization over local unitary operations. For example, let $\mathcal{U} = U_1 \otimes U_2 \otimes \cdots \otimes U_m$, where $U_j \in U(N_j, \mathbf{C})$. Then we maximize the GHZ^m class concurrences for a given pure m-partite state over all local unitary operations $\mathcal{U}$. If $\max_{\forall\mathcal{U}} \mathcal{C}\left(\mathcal{Q}^{W^m}_m(N_1, \ldots, N_m)\right) \neq 0$, then we have a genuine GHZ^m multipartite state by construction. As an example of multi-qubit state let us consider a state

$|W^m\rangle = \frac{1}{\sqrt{m}}(|1, 1, \ldots, 1, 2\rangle + \cdots + |2, 1, \ldots, 1, 1\rangle)$. For this state, the W^m class concurrence is

$$\mathcal{C}\big(\mathcal{Q}_m^{W^m}(2, \ldots, 2)\big) = \left(\frac{4C(m, 2)}{m^2}\mathcal{N}_m^W\right)^{1/2} = \left(\frac{2(m-1)}{m}\mathcal{N}_m^W\right)^{1/2}. \tag{20}$$

This value coincides with that given by Dür [21]. Moreover, let us consider the following state $|\mathrm{GHZ}^m\rangle = \frac{1}{\sqrt{2}}(|1, \ldots, 1\rangle + |2, \ldots, 2\rangle)$. For this state the GHZ^m class concurrence is

$$\mathcal{C}\big(\mathcal{Q}_m^{\mathrm{GHZ}^m}(2, \ldots, 2)\big) = \left(\frac{4C(m, 2)}{4}\mathcal{N}_m^{\mathrm{GHZ}}\right)^{1/2} = \left(\frac{m(m-1)}{2}\mathcal{N}_m^{\mathrm{GHZ}}\right)^{1/2}. \tag{21}$$

We will discuss in detail these states for three-qubit and four-qubit states below. Finally, for some partially separable states the $\mathcal{C}\big(\mathcal{Q}_m^{W^m}(N_1, \ldots, N_m)\big)$ class and $\mathcal{C}\big(\mathcal{Q}_m^{\mathrm{GHZ}^m}(N_1, \ldots, N_m)\big)$ class concurrences do not exactly quantify entanglement in general. Example of such states can be, e.g., constructed for three-qubit states.

5. Concurrence classes for general pure three-partite states

In this section, we will construct concurrences for general pure three-partite states based on orthogonal complement of our POVM. For three-partite states, we have two different joint phases in our POVM, those which are sum and difference of phases of two subsystem, i.e., $\big(\varphi_{\mathcal{Q}_1;k_1,l_1} \pm \varphi_{\mathcal{Q}_2;k_2,l_2}\big)$ and those which are sum and difference of phases of three subsystem, i.e., $\big(\varphi_{\mathcal{Q}_1;k_1,l_1} \pm \varphi_{\mathcal{Q}_2;k_2,l_2} \pm \varphi_{\mathcal{Q}_3;k_3,l_3}\big)$. The first one identifies W^3 class and the second one identifies GHZ^3 class. For the W^3 class concurrence, we have three types of entanglement, entanglement between subsystems one and two $\mathcal{Q}_1\mathcal{Q}_2$, one and three $\mathcal{Q}_1\mathcal{Q}_3$, and two and three $\mathcal{Q}_2\mathcal{Q}_3$. So, we define a linear operator by

$$\widetilde{\Delta}^{W^3_{\Lambda_3}}_{\mathcal{Q}_{1,2}(N_1,N_2)} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1;k_1,l_1}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2;k_2,l_2}\big) \otimes \mathcal{I}_{N_3}.$$

$\widetilde{\Delta}^{W^3_{\Lambda_3}}_{\mathcal{Q}_{1,3}(N_1,N_3)}$ and $\widetilde{\Delta}^{W^3_{\Lambda_3}}_{\mathcal{Q}_{2,3}(N_2,N_3)}$ are defined in the similar way. Now, for pure quantum system $\mathcal{Q}_3^p(N_1, N_2, N_3)$ with

$$\mathcal{C}\big(\mathcal{Q}^{W^3}_{r_1,r_2}(N_{r_1}, N_{r_2})\big) = \sum_{\forall k_1,l_1,k_2,l_2,;k_3,l_3} \left|\big\langle\Psi\big|\widetilde{\Delta}^{W^3_{\Lambda_3}}_{\mathcal{Q}_{r_1,r_2}(N_{r_1},N_{r_2})}\mathcal{C}_3\Psi\big\rangle\right|^2, \tag{22}$$

the W^3 class concurrence is given by

$$\begin{aligned}\mathcal{C}\big(\mathcal{Q}_3^{W^3}(N_1, N_2, N_3)\big) &= \left(\mathcal{N}_3^W \sum_{1=r_1<r_2}^{3} \mathcal{C}\big(\mathcal{Q}^W_{r_1,r_2}(N_{r_1}, N_{r_2})\big)\right)^{1/2} \\ &= \left(4\mathcal{N}_3^W\left[\sum_{l_1>k_1=1}^{N_1}\sum_{l_2>k_2=1}^{N_2}\left|\sum_{k_3=l_3=1}^{N_3}\big(\alpha_{k_1,l_2,k_3}\alpha_{l_1,k_2,l_3} - \alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}\big)\right|^2\right.\right. \\ &\quad + \sum_{l_1>k_1=1}^{N_1}\sum_{l_3>k_3=1}^{N_3}\left|\sum_{k_2=l_2=1}^{N_2}\big(\alpha_{k_1,k_2,l_3}\alpha_{l_1,l_2,k_3} - \alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}\big)\right|^2 \\ &\quad \left.\left. + \sum_{l_2>k_2=1}^{N_2}\sum_{l_3>k_3=1}^{N_3}\left|\sum_{k_1=l_1=1}^{N_1}\big(\alpha_{k_1,k_2,l_3}\alpha_{l_1,l_2,k_3} - \alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}\big)\right|^2\right]\right)^{1/2},\end{aligned} \tag{23}$$

where $\mathcal{N}_3^W$ is a normalization constant. By construction, the W^3 class concurrence for three-partite states vanishes for product states. Now, for a state $|\Psi_{W^3}\rangle = \alpha_{1,1,2}|1,1,2\rangle + \alpha_{1,2,1}|1,2,1\rangle + \alpha_{2,1,1}|2,1,1\rangle$, the W^3 class concurrence gives

$$\mathcal{C}\big(\mathcal{Q}_3^{W^3}(2,2,2)\big) = \big(4\mathcal{N}_3^W[|\alpha_{1,2,1}\alpha_{2,1,1}|^2 + |\alpha_{1,1,2}\alpha_{2,1,1}|^2 + |\alpha_{1,1,2}\alpha_{1,2,1}|^2]\big)^{1/2}.$$

When $\alpha_{1,1,2} = \alpha_{1,2,1} = \alpha_{2,1,1} = \frac{1}{\sqrt{3}}$, we get $\mathcal{C}\big(\mathcal{Q}_3^{W^3}(2,2,2)\big) = \big(\frac{4}{3}\mathcal{N}_3^W\big)^{1/2}$ and $\mathcal{C}\big(\mathcal{Q}_3^{\mathrm{GHZ}^3}(2,2,2)\big) = 0$. Thus, for $\mathcal{N}_3^W = \frac{3}{4}$, we have $\mathcal{C}\big(\mathcal{Q}_3^{W^3}(2,2,2)\big) = 1$.

The second class of three-partite state that we would like to consider is the GHZ^3 class. For this class, we have three types of entanglement, so there are three linear operators. The first one is given by

$$\widetilde{\Delta}^{\mathrm{GHZ}^3_{\Lambda_3}}_{\mathcal{Q}_{1,2}(N_1,N_2)} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1;k_1,l_1}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2;k_2,l_2}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi^{\pi}_{\mathcal{Q}_3;k_3,l_3}\big).$$

$\widetilde{\Delta}^{\mathrm{GHZ}^3_{\Lambda_3}}_{\mathcal{Q}_{1,3}(N_1,N_3)}$ and $\widetilde{\Delta}^{\mathrm{GHZ}^3_{\Lambda_3}}_{\mathcal{Q}_{2,3}(N_2,N_3)}$ are defined in similar way. Now, for a pure quantum system $\mathcal{Q}_3^p(N_1,N_2,N_3)$ with

$$\mathcal{C}\big(\mathcal{Q}^{\mathrm{GHZ}^3}_{r_1,r_2}(N_{r_1},N_{r_2})\big) = \sum_{\forall k_1<l_1,k_2<l_2,;k_3<l_3} \left|\big\langle\Psi\big|\widetilde{\Delta}^{\mathrm{GHZ}^3_{\Lambda_3}}_{\mathcal{Q}_{r_1,r_2}(N_{r_1},N_{r_2})}\mathcal{C}_3\Psi\big\rangle\right|^2, \tag{24}$$

the GHZ^3 class concurrence for general pure three-partite states is given by

$$\begin{aligned}\mathcal{C}\big(\mathcal{Q}_3^{\mathrm{GHZ}^3}(N_1,N_2,N_3)\big) &= \left(\mathcal{N}_3^{\mathrm{GHZ}}\sum_{1=r_1<r_2}^{3}\mathcal{C}\big(\mathcal{Q}^{\mathrm{GHZ}}_{r_1,r_2}(N_{r_1},N_{r_2})\big)\right)^{1/2}\\
&= \Bigg(4\mathcal{N}_3^{\mathrm{GHZ}}\Bigg[\sum_{l_1>k_1}^{N_1}\sum_{k_1=1}^{N_1}\sum_{l_2>k_2}^{N_2}\sum_{k_2=1}^{N_2}\sum_{l_3>k_3}^{N_3}\sum_{k_3=1}^{N_3}\big|\alpha_{k_1,l_2,l_3}\alpha_{l_1,k_2,k_3}\\
&\quad+\alpha_{k_1,l_2,k_3}\alpha_{l_1,k_2,l_3}-\alpha_{k_1,k_2,l_3}\alpha_{l_1,l_2,k_3}-\alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}\big|^2+\big|\alpha_{k_1,l_2,l_3}\alpha_{l_1,k_2,k_3}\\
&\quad-\alpha_{k_1,l_2,k_3}\alpha_{l_1,k_2,l_3}+\alpha_{k_1,k_2,l_3}\alpha_{l_1,l_2,k_3}-\alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}\big|^2+\big|-\alpha_{k_1,l_2,l_3}\alpha_{l_1,k_2,k_3}\\
&\quad+\alpha_{k_1,l_2,k_3}\alpha_{l_1,k_2,l_3}+\alpha_{k_1,k_2,l_3}\alpha_{l_1,l_2,k_3}-\alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3}\big|^2\Bigg]\Bigg)^{1/2},\end{aligned} \tag{25}$$

where $\mathcal{N}_3^{\mathrm{GHZ}}$ is a normalization constant. Now, for the state $|\Psi_{\mathrm{GHZ}^3}\rangle = \alpha_{1,1,1}|1,1,1\rangle + \alpha_{2,2,2}|2,2,2\rangle$, the GHZ^3 class concurrence gives $\mathcal{C}\big(\mathcal{Q}_3^{\mathrm{GHZ}^3}(2,2,2)\big) = \big(12\mathcal{N}_3^{\mathrm{GHZ}}|\alpha_{1,1,1}\alpha_{2,2,2}|^2\big)^{1/2}$ and for $\alpha_{1,1,1} = \alpha_{2,2,2} = \frac{1}{\sqrt{2}}$, we get $\mathcal{C}\big(\mathcal{Q}_3^{\mathrm{GHZ}^3}(2,2,2)\big) = \big(3\mathcal{N}_3^{\mathrm{GHZ}}\big)^{1/2}$. Thus, for $\mathcal{N}_3^{\mathrm{GHZ}} = \frac{1}{3}$ we have $\mathcal{C}\big(\mathcal{Q}_3^{\mathrm{GHZ}^3}(2,2,2)\big) = 1$. However, for this state $\mathcal{C}\big(\mathcal{Q}_3^{W^3}(2,2,2)\big) = 0$. Note that for some states we need to perform optimization over local unitary operations as was discussed in the previous section.

6. Concurrence classes for general pure four-partite states

For general four-partite states, we have three different joint phases in our POVM. Those which are sum and difference of phases of two subsystems, i.e., $\big(\varphi_{\mathcal{Q}_1;k_1,l_1} \pm \varphi_{\mathcal{Q}_2;k_2,l_2}\big)$, those which are sum and difference of phases of three subsystems, i.e., $\big(\varphi_{\mathcal{Q}_1;k_1,l_1} \pm \varphi_{\mathcal{Q}_2;k_2,l_2} \pm \varphi_{\mathcal{Q}_3;k_3,l_3}\big)$, and those which are sum and difference of phases of four subsystems, i.e., $\big(\varphi_{\mathcal{Q}_1;k_1,l_1} \pm \varphi_{\mathcal{Q}_2;k_2,l_2} \pm \varphi_{\mathcal{Q}_3;k_3,l_3} \pm \varphi_{\mathcal{Q}_4;k_4,l_4}\big)$. The first one identifies W^4 class concurrence, the second one identifies GHZ^3 class concurrence, and the third one identifies GHZ^4 class concurrence. For the

W^4 class, we have six types of entanglement, so there are six operators corresponding to entanglement between $\mathcal{Q}_1\mathcal{Q}_2$, $\mathcal{Q}_1\mathcal{Q}_3$, $\mathcal{Q}_1\mathcal{Q}_4$, $\mathcal{Q}_2\mathcal{Q}_3$, $\mathcal{Q}_2\mathcal{Q}_4$ and $\mathcal{Q}_3\mathcal{Q}_4$ subsystems. The linear operator corresponding to $\mathcal{Q}_1\mathcal{Q}_2$ is given by

$$\widetilde{\Delta}^{W^4_{\Lambda_4}}_{\mathcal{Q}_{1,2}(N_1,N_2)} = \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1;k_1,l_1}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2;k_2,l_2}\right) \otimes \mathcal{I}_{N_3} \otimes \mathcal{I}_{N_4}.$$

$\widetilde{\Delta}^{W^4_{\Lambda_4}}_{\mathcal{Q}_{1,3}(N_1,N_3)}$, $\widetilde{\Delta}^{W^4_{\Lambda_4}}_{\mathcal{Q}_{1,4}(N_1,N_4)}$, $\widetilde{\Delta}^{W^4_{\Lambda_4}}_{\mathcal{Q}_{2,3}(N_2,N_3)}$, $\widetilde{\Delta}^{W^4_{\Lambda_4}}_{\mathcal{Q}_{2,4}(N_2,N_4)}$ and $\widetilde{\Delta}^{W^4_{\Lambda_4}}_{\mathcal{Q}_{3,4}(N_2,N_4)}$ are defined in similar way. Now, for a pure quantum system $\mathcal{Q}^p_4(N_1,\ldots,N_4)$ with

$$\mathcal{C}\left(\mathcal{Q}^{W^4}_{r_1,r_2}\left(N_{r_1},N_{r_2}\right)\right) = \sum_{\forall k_1,l_1,\ldots,k_4,l_4} \left|\left\langle\Psi\right|\widetilde{\Delta}^{W^4_{\Lambda_4}}_{\mathcal{Q}_{r_1,r_2}(N_{r_1},N_{r_2})}\mathcal{C}_4\Psi\right\rangle\right|^2, \tag{26}$$

the W^4 class concurrence is given by

$$\begin{aligned}
\mathcal{C}\left(\mathcal{Q}^{W^4}_4(N_1,\ldots,N_4)\right) &= \left(\mathcal{N}^W_4 \sum_{1=r_1<r_2}^{4} \mathcal{C}\left(\mathcal{Q}^{W^4}_{r_1,r_2}\left(N_{r_1},N_{r_2}\right)\right)\right)^{1/2} \\
&= \left(4\mathcal{N}^W_4 \sum_{l_1>k_1=1}^{N_1}\sum_{l_2>k_2=1}^{N_2}\left|\sum_{k_3=l_3=1}^{N_3}\sum_{k_4=l_4=1}^{N_4}\left(\alpha_{k_1,l_2,k_3,k_4}\alpha_{l_1,k_2,l_3,l_4} - \alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4}\right)\right|^2\right. \\
&\quad + \sum_{l_1>k_1=1}^{N_1}\sum_{l_3>k_3=1}^{N_3}\left|\sum_{k_2=l_2=1}^{N_2}\sum_{k_4=l_4=1}^{N_4}\left(\alpha_{k_1,k_2,l_3,k_4}\alpha_{l_1,l_2,k_3,l_4} - \alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4}\right)\right|^2 \\
&\quad + \sum_{l_1>k_1=1}^{N_1}\sum_{l_4>k_4=1}^{N_4}\left|\sum_{k_2=l_2=1}^{N_2}\sum_{k_3=l_3=1}^{N_3}\left(\alpha_{k_1,k_2,k_3,l_4}\alpha_{l_1,l_2,l_3,k_4} - \alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4}\right)\right|^2 \\
&\quad + \sum_{l_2>k_2=1}^{N_2}\sum_{l_3>k_3=1}^{N_3}\left|\sum_{k_1=l_1=1}^{N_1}\sum_{k_4=l_4=1}^{N_4}\left(\alpha_{k_1,k_2,l_3,k_4}\alpha_{l_1,l_2,k_3,l_4} - \alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4}\right)\right|^2 \\
&\quad + \sum_{l_2>k_2=1}^{N_2}\sum_{l_4>k_4=1}^{N_4}\left|\sum_{k_1=l_1=1}^{N_1}\sum_{k_3=l_3=1}^{N_3}\left(\alpha_{k_1,k_2,k_3,l_4}\alpha_{l_1,l_2,l_3,k_4} - \alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4}\right)\right|^2 \\
&\quad \left. + \sum_{l_3>k_3=1}^{N_3}\sum_{l_4>k_4=1}^{N_4}\left|\sum_{k_1=l_1=1}^{N_1}\sum_{k_2=l_2=1}^{N_2}\left(\alpha_{k_1,k_2,k_3,l_4}\alpha_{l_1,l_2,l_3,k_4} - \alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4}\right)\right|^2\right)^{1/2},
\end{aligned} \tag{27}$$

where $\mathcal{N}^W_4$ is a normalization constant. Now, for a state $|\Psi_{W^4}\rangle = \alpha_{1,1,1,2}|1,1,1,2\rangle + \alpha_{1,1,2,1}|1,1,2,1\rangle + \alpha_{1,2,1,1}|1,2,1,1\rangle + \alpha_{2,1,1,1}|2,1,1,1\rangle$, the W^4 class concurrence gives

$$\begin{aligned}
\mathcal{C}\left(\mathcal{Q}^{W^4}_4(2,\ldots,2)\right) = &\left(4\mathcal{N}^W_3[|\alpha_{1,2,1,1}\alpha_{2,1,1,1}|^2 + |\alpha_{1,1,2,1}\alpha_{2,1,1,1}|^2 + |\alpha_{1,1,1,2}\alpha_{2,1,1,1}|^2\right. \\
&\left. + |\alpha_{1,1,2,1}\alpha_{1,2,1,1}|^2 + |\alpha_{1,1,1,2}\alpha_{1,2,1,1}|^2 + |\alpha_{1,1,1,2}\alpha_{1,1,2,1}|^2]\right)^{1/2}
\end{aligned}$$

and for $\alpha_{1,1,1,2} = \alpha_{1,1,2,1} = \alpha_{1,2,1,1} = \alpha_{1,2,1,1} = \frac{1}{\sqrt{4}}$, we get $\mathcal{C}\left(\mathcal{Q}^{W^4}_4(2,\ldots,2)\right) = \left(\frac{4C(4,2)}{4^2}\mathcal{N}^W_4\right)^{1/2} = \left(\frac{3}{2}\mathcal{N}^W_4\right)^{1/2}$, $\mathcal{C}\left(\mathcal{Q}^{\mathrm{GHZ}^3}_4(2,\ldots,2)\right) = 0$.

The second class of four-partite state that we want to consider is the GHZ^3 class. For this class, we have four types of entanglement. These linear operators are given by

$$\begin{aligned}
\widetilde{\Delta}^{\mathrm{GHZ}^3_{\Lambda_4}}_{\mathcal{Q}_{12,3}(N_1,N_2)} &= \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1;k_1,l_1}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2;k_2,l_2}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\left(\varphi^{\pi}_{\mathcal{Q}_3;k_3,l_3}\right) \otimes \mathcal{I}_{N_4}, \\
\widetilde{\Delta}^{\mathrm{GHZ}^3_{\Lambda_4}}_{\mathcal{Q}_{12,4}(N_1,N_3)} &= \widetilde{\Delta}_{\mathcal{Q}_1}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1;k_1,l_1}\right) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\left(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2;k_2,l_2}\right) \otimes \mathcal{I}_{N_3} \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\left(\varphi^{\pi}_{\mathcal{Q}_4;k_4,l_4}\right),
\end{aligned}$$

$$\widetilde{\Delta}^{\mathrm{GHZ}^3_{\Lambda_4}}_{\mathcal{Q}_{13,4}(N_1,N_4)} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1;k_1,l_1}\big) \otimes \mathcal{I}_{N_2} \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_3;k_3,l_3}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi^{\pi}_{\mathcal{Q}_4;k_4,l_4}\big),$$
$$\widetilde{\Delta}^{\mathrm{GHZ}^3_{\Lambda_4}}_{\mathcal{Q}_{23,4}(N_2,N_3)} = \mathcal{I}_{N_1} \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2;k_2,l_2}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_3;k_3,l_3}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi^{\pi}_{\mathcal{Q}_4;k_4,l_4}\big),$$

where, e.g., $\widetilde{\Delta}^{\mathrm{GHZ}^3_{\Lambda_4}}_{\mathcal{Q}_{12,4}(N_1,N_2)}$ identifies elements of our POVM which are sum and difference of phases of three subsystems, i.e., $\big(\big[\varphi_{\mathcal{Q}_1;k_1,l_1} \pm \varphi_{\mathcal{Q}_2;k_2,l_2}\big] \pm \varphi_{\mathcal{Q}_4;k_4,l_4}\big)$ and extract information about entanglement of subsystems $\mathcal{Q}_1\mathcal{Q}_2 - \mathcal{Q}_4$. For pure quantum system $\mathcal{Q}^p_4(N_1,\ldots,N_4)$ with

$$\mathcal{C}\big(\mathcal{Q}^{\mathrm{GHZ}^3}_{r_1r_2,r_3}(N_{r_1},N_{r_2})\big) = \sum_{\forall k_1,l_1,\ldots,k_4,l_4} \big|\big\langle\Psi\big|\widetilde{\Delta}^{\mathrm{GHZ}^3_{\Lambda_4}}_{\mathcal{Q}_{r_1r_2,r_3}(N_{r_1},N_{r_2})}\mathcal{C}_4\Psi\big\rangle\big|^2, \tag{28}$$

the GHZ^3 class concurrence for general four-partite state is given by

$$\begin{aligned}
\mathcal{C}\big(\mathcal{Q}^{\mathrm{GHZ}^3}_4(N_1,\ldots,N_4)\big) &= \left(\mathcal{N}^{\mathrm{GHZ}}_3 \sum_{1=r_1<r_2<r_3}^{4} \mathcal{C}\big(\mathcal{Q}^{\mathrm{GHZ}^3_{\Lambda_4}}_{r_1r_2,r_3}(N_{r_1},N_{r_2})\big)\right)^{1/2} \\
&= \Bigg(\mathcal{N}^{\mathrm{GHZ}}_3 \sum_{l_1>k_1=1}^{N_1}\sum_{l_2>k_2=1}^{N_2}\sum_{l_3>k_3=1}^{N_3}\Bigg|\sum_{k_4=l_4=1}^{N_4}\big(-\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4} \\
&\quad - \alpha_{k_1,k_2,l_3,k_4}\alpha_{l_1,l_2,k_3,l_4} + \alpha_{k_1,l_2,k_3,k_4}\alpha_{l_1,k_2,l_3,l_4} + \alpha_{k_1,l_2,l_3,k_4}\alpha_{l_1,k_2,k_3,l_4}\big)\Bigg|^2 \\
&\quad + \sum_{l_1>k_1=1}^{N_1}\sum_{l_2>k_2=1}^{N_2}\sum_{l_4>k_4=1}^{N_4}\Bigg|\sum_{k_3=l_3=1}^{N_3}\big(-\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4} - \alpha_{k_1,k_2,k_3,l_4}\alpha_{l_1,l_2,l_3,k_4} \\
&\quad + \alpha_{k_1,l_2,k_3,k_4}\alpha_{l_1,k_2,l_3,l_4} + \alpha_{k_1,l_2,k_3,l_4}\alpha_{l_1,k_2,l_3,k_4}\big)\Bigg|^2 \\
&\quad + \sum_{l_1>k_1=1}^{N_1}\sum_{l_3>k_3=1}^{N_3}\sum_{l_4>k_4=1}^{N_4}\Bigg|\sum_{k_2=l_2=1}^{N_2}\big(-\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4} - \alpha_{k_1,k_2,k_3,l_4}\alpha_{l_1,l_2,l_3,k_4} \\
&\quad + \alpha_{k_1,k_2,l_3,k_4}\alpha_{l_1,l_2,k_3,l_4} + \alpha_{k_1,k_2,l_3,l_4}\alpha_{l_1,l_2,k_3,k_4}\big)\Bigg|^2 \\
&\quad + \sum_{l_2>k_2=1}^{N_2}\sum_{l_3>k_3=1}^{N_3}\sum_{l_4>k_4=1}^{N_4}\Bigg|\sum_{k_1=l_1=1}^{N_1}\big(-\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4} + \alpha_{k_1,k_2,k_3,l_4}\alpha_{l_1,l_2,l_3,k_4} \\
&\quad - \alpha_{k_1,k_2,l_3,k_4}\alpha_{l_1,l_2,k_3,l_4} + \alpha_{k_1,k_2,l_3,l_4}\alpha_{l_1,l_2,k_3,k_4}\Bigg|^2\Bigg)^{1/2},
\end{aligned}$$

where $\mathcal{N}^{\mathrm{GHZ}}_3$ is a normalization constant. Next we are going to consider the GHZ^4 class concurrence for general four-partite states. For the GHZ^4 class, we have again six types of entanglement, so there are six linear operators corresponding to entanglement between these subsystems. The linear operator corresponding to $(\mathcal{Q}_1\mathcal{Q}_2)\mathcal{Q}_3\mathcal{Q}_4$ is given by

$$\widetilde{\Delta}^{\mathrm{GHZ}^4_{\Lambda_4}}_{\mathcal{Q}_{1,2}(N_1,N_2)} = \widetilde{\Delta}_{\mathcal{Q}_1}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_1;k_1,l_1}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_2}\big(\varphi^{\frac{\pi}{2}}_{\mathcal{Q}_2;k_2,l_2}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_3}\big(\varphi^{\pi}_{\mathcal{Q}_3;k_3,l_3}\big) \otimes \widetilde{\Delta}_{\mathcal{Q}_4}\big(\varphi^{\pi}_{\mathcal{Q}_4;k_4,l_4}\big), \tag{29}$$

$\widetilde{\Delta}^{\mathrm{GHZ}^4_{\Lambda_4}}_{\mathcal{Q}_{1,3}(N_1,N_3)}$, $\widetilde{\Delta}^{\mathrm{GHZ}^4_{\Lambda_4}}_{\mathcal{Q}_{1,4}(N_1,N_4)}$, $\widetilde{\Delta}^{\mathrm{GHZ}^4_{\Lambda_4}}_{\mathcal{Q}_{2,3}(N_2,N_3)}$, $\widetilde{\Delta}^{\mathrm{GHZ}^4_{\Lambda_4}}_{\mathcal{Q}_{2,4}(N_2,N_4)}$, and $\widetilde{\Delta}^{\mathrm{GHZ}^4_{\Lambda_4}}_{\mathcal{Q}_{3,4}(N_3,N_4)}$ are defined in a similar way. Now, for a pure quantum system $\mathcal{Q}^p_4(N_1, \ldots, N_4)$, let

$$\mathcal{C}\big(\mathcal{Q}^{\mathrm{GHZ}^4}_{r_1,r_2}(N_{r_1}, N_{r_2})\big) = \sum_{\forall k_1<l_1,\ldots,k_4<l_4} \left|\langle\Psi\right|\widetilde{\Delta}^{\mathrm{GHZ}^4_{\Lambda_4}}_{\mathcal{Q}_{r_1,r_2}(N_{r_1},N_{r_2})}\mathcal{C}_4\Psi\rangle\big|^2. \tag{30}$$

Then, the GHZ^4 class concurrence is given by

$$\begin{aligned}
\mathcal{C}\big(\mathcal{Q}^{\mathrm{GHZ}^4}_4(N_1, \ldots, N_4)\big) &= \left(\mathcal{N}^{\mathrm{GHZ}}_4 \sum_{1=r_1<r_2}^{4} \mathcal{C}\big(\mathcal{Q}^{\mathrm{GHZ}^4}_{r_1,r_2}(N_{r_1}, N_{r_2})\big)\right)^{1/2} \\
&= \Bigg(4\mathcal{N}^{\mathrm{GHZ}}_4 \sum_{l_1>k_1=1}^{N_1}\sum_{l_2>k_2=1}^{N_2}\sum_{l_3>k_3=1}^{N_3}\sum_{l_4>k_4=1}^{N_4} \big[\big|-\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4} \\
&\quad - \alpha_{k_1,k_2,k_3,l_4}\alpha_{l_1,l_2,l_3,k_4} - \alpha_{k_1,k_2,l_3,k_4}\alpha_{l_1,l_2,k_3,l_4} - \alpha_{k_1,k_2,l_3,l_4}\alpha_{l_1,l_2,k_3,k_4} \\
&\quad + \alpha_{k_1,l_2,k_3,k_4}\alpha_{l_1,k_2,l_3,l_4} + \alpha_{k_1,l_2,k_3,l_4}\alpha_{l_1,k_2,l_3,k_4} + \alpha_{k_1,l_2,l_3,k_4}\alpha_{l_1,k_2,k_3,l_4} \\
&\quad + \alpha_{k_1,l_2,l_3,l_4}\alpha_{l_1,k_2,k_3,k_4}\big|^2 + \big|-\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4} - \alpha_{k_1,k_2,k_3,l_4}\alpha_{l_1,l_2,l_3,k_4} \\
&\quad + \alpha_{k_1,k_2,l_3,k_4}\alpha_{l_1,l_2,k_3,l_4} + \alpha_{k_1,k_2,l_3,l_4}\alpha_{l_1,l_2,k_3,k_4} - \alpha_{k_1,l_2,k_3,k_4}\alpha_{l_1,k_2,l_3,l_4} \\
&\quad - \alpha_{k_1,l_2,k_3,l_4}\alpha_{l_1,k_2,l_3,k_4} + \alpha_{k_1,l_2,l_3,k_4}\alpha_{l_1,k_2,k_3,l_4} + \alpha_{k_1,l_2,l_3,l_4}\alpha_{l_1,k_2,k_3,k_4}\big|^2 \\
&\quad + \big|-\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4} + \alpha_{k_1,k_2,k_3,l_4}\alpha_{l_1,l_2,l_3,k_4} - \alpha_{k_1,k_2,l_3,k_4}\alpha_{l_1,l_2,k_3,l_4} \\
&\quad + \alpha_{k_1,k_2,l_3,l_4}\alpha_{l_1,l_2,k_3,k_4} - \alpha_{k_1,l_2,k_3,k_4}\alpha_{l_1,k_2,l_3,l_4} + \alpha_{k_1,l_2,k_3,l_4}\alpha_{l_1,k_2,l_3,k_4} \\
&\quad + \alpha_{k_1,l_2,l_3,k_4}\alpha_{l_1,k_2,k_3,l_4} - \alpha_{k_1,l_2,l_3,l_4}\alpha_{l_1,k_2,k_3,k_4}\big|^2 + \big|-\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4} \\
&\quad - \alpha_{k_1,k_2,k_3,l_4}\alpha_{l_1,l_2,l_3,k_4} + \alpha_{k_1,k_2,l_3,k_4}\alpha_{l_1,l_2,k_3,l_4} + \alpha_{k_1,k_2,l_3,l_4}\alpha_{l_1,l_2,k_3,k_4} \\
&\quad + \alpha_{k_1,l_2,k_3,k_4}\alpha_{l_1,k_2,l_3,l_4} + \alpha_{k_1,l_2,k_3,l_4}\alpha_{l_1,k_2,l_3,k_4} - \alpha_{k_1,l_2,l_3,k_4}\alpha_{l_1,k_2,k_3,l_4} \\
&\quad - \alpha_{k_1,l_2,l_3,l_4}\alpha_{l_1,k_2,k_3,k_4}\big|^2 + \big|-\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4} + \alpha_{k_1,k_2,k_3,l_4}\alpha_{l_1,l_2,l_3,k_4} \\
&\quad - \alpha_{k_1,k_2,l_3,k_4}\alpha_{l_1,l_2,k_3,l_4} + \alpha_{k_1,k_2,l_3,l_4}\alpha_{l_1,l_2,k_3,k_4} + \alpha_{k_1,l_2,k_3,k_4}\alpha_{l_1,k_2,l_3,l_4} \\
&\quad - \alpha_{k_1,l_2,k_3,l_4}\alpha_{l_1,k_2,l_3,k_4} + \alpha_{k_1,l_2,l_3,k_4}\alpha_{l_1,k_2,k_3,l_4} - \alpha_{k_1,l_2,l_3,l_4}\alpha_{l_1,k_2,k_3,k_4}\big|^2 \\
&\quad + \big|-\alpha_{k_1,k_2,k_3,k_4}\alpha_{l_1,l_2,l_3,l_4} + \alpha_{k_1,k_2,k_3,l_4}\alpha_{l_1,l_2,l_3,k_4} + \alpha_{k_1,k_2,l_3,k_4}\alpha_{l_1,l_2,k_3,l_4} \\
&\quad - \alpha_{k_1,k_2,l_3,l_4}\alpha_{l_1,l_2,k_3,k_4} - \alpha_{k_1,l_2,k_3,k_4}\alpha_{l_1,k_2,l_3,l_4} + \alpha_{k_1,l_2,k_3,l_4}\alpha_{l_1,k_2,l_3,k_4} \\
&\quad + \alpha_{k_1,l_2,l_3,k_4}\alpha_{l_1,k_2,k_3,l_4} - \alpha_{k_1,l_2,l_3,l_4}\alpha_{l_1,k_2,k_3,k_4}\big|^2\big]\Bigg)^{1/2},
\end{aligned} \tag{31}$$

where $\mathcal{N}^{\mathrm{GHZ}}_4$ is a normalization constant. As an example, let us investigate the concurrence for the GHZ^3 class of four-qubit state. Let $\beta_1 = \alpha_{1,1,1,1}\alpha_{2,2,2,2}$, $\beta_2 = \alpha_{1,1,1,2}\alpha_{2,2,2,1}$, $\beta_3 = \alpha_{1,1,2,1}\alpha_{2,2,1,2}$, $\beta_4 = \alpha_{1,1,2,2}\alpha_{2,2,1,1}$, $\beta_5 = \alpha_{1,2,1,1}\alpha_{2,1,2,2}$, $\beta_6 = \alpha_{1,2,1,2}\alpha_{2,1,2,1}$, $\beta_7 = \alpha_{1,2,2,1}\alpha_{2,1,1,2}$, $\beta_8 = \alpha_{1,2,2,2}\alpha_{2,1,1,1}$, then we have

$$\begin{aligned}
\mathcal{C}\big(4\mathcal{Q}^{\mathrm{GHZ}^4}_4(2, \ldots, 2)\big) = \big(4\mathcal{N}^{\mathrm{GHZ}}_4&\big[|-\beta_1 - \beta_2 - \beta_3 - \beta_4 + \beta_5 + \beta_6 \\
&+ \beta_7 + \beta_8|^2 + |-\beta_1 - \beta_2 + \beta_3 + \beta_4 - \beta_5 - \beta_6 + \beta_7 + \beta_8|^2 \\
&+ |-\beta_1 + \beta_2 - \beta_3 + \beta_4 - \beta_5 + \beta_6 + \beta_7 - \beta_8|^2 + |-\beta_1 - \beta_2 \\
&+ \beta_3 + \beta_4 + \beta_5 + \beta_6 - \beta_7 - \beta_8|^2 + |-\beta_1 + \beta_2 - \beta_3 + \beta_4 + \beta_5 \\
&- \beta_6 + \beta_7 - \beta_8|^2 + |-\beta_1 + \beta_2 + \beta_3 - \beta_4 - \beta_5 + \beta_6 + \beta_7 - \beta_8|^2\big]\big)^{1/2}.
\end{aligned}$$

Next, let us consider following the state $\left|\Psi^4_{\mathrm{GHZ}}\right\rangle = \alpha_{1,1,1,1}|1, 1, 1, 1\rangle + \alpha_{2,2,2,2}|2, 2, 2, 2\rangle$ then the concurrence of the GHZ^4 class of the $\left|\Psi^4_{\mathrm{GHZ}}\right\rangle$ state is given by

$$\mathcal{C}\big(\mathcal{Q}_4^{\mathrm{GHZ}^4}(2,\ldots,2)\big) = \big(4\cdot 6\mathcal{N}_4^{\mathrm{GHZ}}|\alpha_{1,1,1,1}\alpha_{2,2,2,2}|^2\big)^{1/2} \tag{32}$$

and for $\alpha_{1,1,1,1} = \alpha_{2,2,2,2} = \frac{1}{\sqrt{2}}$ we get $\mathcal{C}\big(\mathcal{Q}_4^{\mathrm{GHZ}^4}(2,\ldots,2)\big) = \big(6\mathcal{N}_4^{\mathrm{GHZ}}\big)^{1/2}$.

7. Conclusion

In this paper, we have expressed concurrence for a general pure bipartite state based on an orthogonal complement of our POVM. Moreover, we have proposed different concurrence classes for pure multipartite states. We have investigate the monotonicity of the W^m class and the GHZ^m class concurrences for multi-qubit states. The W^m class concurrence for multi-qubit states are entanglement monotones. However, the GHZ^m class concurrences are not entanglement monotones. Our classification suggested the existence different classes of multipartite entanglement which are in equivalent under LOCC. At least, we know that there are two different classes of entanglement for multi-qubit states which our methods could distinguish very well. For higher dimensional composite states, e.g., for m-partite states for $m \geqslant 4$, there is no well-known and well-accepted classification. Thus, there is more space for new idea and methods to give a rigorous classification of multipartite states. However, we think that this work is a timely contribution to the relatively large effort presently being undertaken to quantify and classify multipartite entanglement.

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